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# Note

## A Sperner Theorem on Unrelated Chains of Subsets

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A theorem of Sperner [2] states that a collection of subsets of $\{1,..., n\}$, no two ordered by inclusion, contains at most $\binom{n}{\lfloor n/2 \rfloor}$ sets. How many two-element chains $A \subset B$ of subsets of $\{1,..., n\}$ can be found such that sets in different chains are not related? More generally, we seek to determine $f_k(n)$, defined to be the maximum $m$ such that there exist subsets $A(i,j) \subseteq \{1,..., n\}$, $1 \leqslant i \leqslant m$, $0 \leqslant j \leqslant k$, satisfying

$$\text{for all } i,\ A(i,0) \subset A(i,1) \subset \cdots \subset A(i,k) \tag{1}$$

and

$$\text{for all } i, i', j, j', \text{ with } i \neq i',\ A(i,j) \not\subseteq A(i',j'). \tag{2}$$

We can obtain such a collection of $m = \binom{n-k}{\lfloor (n-k)/2 \rfloor}$ unrelated chains of $k+1$ sets each as follows: The sets $A(i,0)$ are the $\lfloor (n-k)/2 \rfloor$ – subsets of $\{k+1,..., n\}$, and for $j \geqslant 1$, $A(i,j) = A(i,0) \cup \{1,...,j\}$. In fact this $m$ is best-possible for all $k \geqslant 0$, which will follow from this generalization of Lubell's inequality [1].

THEOREM 1. *Suppose $A_1 \subseteq B_1,..., A_m \subseteq B_m$ are subsets of $\{1,..., n\}$ such that $A_i \not\subseteq B_{i'}$, for $i \neq i'$.*

*Then*

$$\sum_{i=1}^{m} \frac{1}{\binom{n-|B_i - A_i|}{|A_i|}} \leqslant 1.$$

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*Proof.* A maximal chain of subsets is of the form

$$\phi = S_0 \subset S_1 \subset \cdots \subset S_n = \{1,\ldots, n\}.$$

The chain is formed by adding one element at a time in some order. When does such a chain intersect an interval $[A_i, B_i] = \{C | A_i \subseteq C \subseteq B_i\}$? They intersect if and only if all elements of $A_i$ are added to the chain before any elements outside $B_i$ are added to the chain. There are $n - |B_i - A_i|$ elements which are either in $A_i$ or not in $B_i$. The orders in which these elements are added to the chains are equally likely. The proportion of maximal chains which intersect $[A_i, B_i]$ is thus $1/\binom{n-|B_i-A_i|}{|A_i|}$. No chain intersects more than one of the intervals $[A_i, B_i]$ because if, say, $S_a \subset S_b$ and $S_a \in [A_i, B_i]$ and $S_b \in [A_j, B_j]$ then $A_i \subseteq B_j$ which implies $i = j$. The sum of these proportions is then at most 1, which is the desired inequality. ∎

Lubell's inequality is obtained for antichains $\{A_1,\ldots, A_m\}$ by taking $B_i = A_i$ for all $i$. We now determine $f_k(n)$. This reduces to Sperner's theorem for $k = 0$.

THEOREM 2. $f_k(n) = \binom{n-k}{\lfloor (n-k)/2 \rfloor}$.

*Proof.* For given $k$ and $n$, let $A(i,j)$ be a collection of $m = f_k(n)$ chains of subsets $A(i,j)$ satisfying (1) and (2). Let $A_i = A(i, 0)$, $B_i = A(i, k)$. Then,

$$\binom{n - |B_i - A_i|}{|A_i|} \leqslant \binom{n-k}{|A_i|} \leqslant \binom{n-k}{\lfloor (n-k)/2 \rfloor}.$$

Hence,

$$\begin{aligned} f_k(n) &= \sum_{i=1}^{m} 1 \\ &\leqslant \sum_{i=1}^{m} \left( \binom{n-k}{\lfloor (n-k)/2 \rfloor} \Big/ \binom{n - |B_i - A_i|}{|A_i|} \right) \\ &\leqslant \binom{n-k}{\lfloor (n-k)/2 \rfloor}, \end{aligned}$$

by the inequality in Theorem 1 which applies to these $A_i$ and $B_i$. The theorem follows by the contruction above of a collection with $\binom{n-k}{\lfloor (n-k)/2 \rfloor}$ chains. ∎

The problem which motivated this study was to determine the 2-dimension of a union of two-element chains [3]. Theorem 2 above implies the solution to this problem, stated in Theorem 3, and generalized to the union of chains with any number of elements. $\underline{k}$ denotes a chain with $k$ elements. $\underline{2}^n$, the product of $n$ copies of $\underline{2}$, is isomorphic to the lattice of subsets of $\{1,\ldots,n\}$. $\dim_2(P)$, the 2-dimension of $P$, is the smallest $n$ such that $P$ can be embedded in $\underline{2}^n$ [4]. $mP$ denotes the disjoint union of $m$ copies of $P$. We can determine $\dim_2(P)$ not just for $P$ a union of $m$ $(k+1)$-chains, but also for a union of $m$ copies of $\underline{2}^k$, because two chains in $\underline{2}^n$ are unrelated if and only if the full intervals with the same tops and bottoms are unrelated.

THEOREM 3. *For $k \geqslant 0$ and $m \geqslant 1$,*

$$\begin{aligned} \dim_2(m(\underline{k+1})) &= \dim_2(m(\underline{2}^k)) \\ &= \min\left\{ n \,\middle|\, \binom{n-k}{\lfloor (n-k)/2 \rfloor} \geqslant m \right\}. \end{aligned}$$

*Remarks.* 1. Sperner's theorem acually says more than Theorem 2 restricted to $k=0$. It states that the only antichain(s) of maximum size in $\underline{2}^n$ are the collection of all subsets of size $\lfloor n/2 \rfloor$ and, for odd $n$, the collection of all subsets of size $\lceil n/2 \rceil$. We conjecture that for general $k$, the only maximum-sized collections of chains are obtained in this natural way: The $A_i$'s consist of all $\lfloor (n-k)/2 \rfloor$-subsets of some $(n-k)$-set (or all $\lceil (n-k/2 \rceil$-subsets), and each $B_i$ equals $A_i$ with the remaining $k$ elements added. The chains can be completed between $A_i$ and $B_i$ in any fashion. Theorem 1 implies that in any maximum-sized collection, each $|A_i|$ equals $\lfloor (n-k)/2 \rfloor$ or $\lceil (n-k)/2 \rceil$ (but not necessarily all $|A_i|$ are equal), and that $|B_i - A_i| = k$ for all $i$.

2. Theorem 1 induces a lower bound on the 2-dimension of a union of chains of varying length. Although the bound is sharp when all chains have the same length, this is not true in general. For instance, if $P$ is a union of $\underline{1}$, $\underline{2}$, and $\underline{3}$, $\dim_2(P) = 5$, yet the inequality of Theorem 1 works for $n = 4$, with $|A_1| = |B_1| = 2$, $|A_2| = 1$, $|B_2| = 2$, $|A_3| = 1$, $|B_3| = 3$.

3. Determining the $t$-dimension of $P$ (i.e., the minimum $n$ such that $P$ can be embedded in $\underline{t}^n$), for $P$ a union of chains seems to be much more difficult when $t > 2$. For the problem of finding the largest size $m$ of a collection of $t$-chains in $\underline{t}^n$ we conjecture that a result similar to Theorem 3 holds: $m$ should be given as the size of the largest antichain in $\underline{t}^{n-1}$ $(n \geqslant 1)$. The general problem of determining the maximal size of a union of $k$-chains that can be embedded in $\underline{t}^n$ for $k+1 \leqslant t$ appears to be totally open.

4. The arguments here can be adapted to prove an inequality for the lattice of subspaces of a finite vector space which is analogous to Theorem 1 for the lattice of subsets.

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